

# Grauert's theorem for subanalytic open sets in real analytic manifolds

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**Abstract**

By open neighbourhood of an open subset  $\Omega$  of  $\mathbb{R}^n$  we mean an open subset  $\Omega'$  of  $\mathbb{C}^n$  such that  $\mathbb{R}^n \cap \Omega' = \Omega$ . A well known result of H. Grauert implies that any open subset of  $\mathbb{R}^n$  admits a fundamental system of Stein open neighbourhoods in  $\mathbb{C}^n$ . Another way to state this property is to say that each open subset of  $\mathbb{R}^n$  is Stein.

We shall prove a similar result in the subanalytic category, so, under the assumption that  $\Omega$  is a subanalytic open subset in a paracompact real analytic manifold, we show that  $\Omega$  admits a fundamental system of subanalytic Stein open neighbourhoods in any of its complexifications.

**1 Introduction.**

A classical result of H. Grauert gives that an open set in a real analytic manifold  $M_{\mathbb{R}}$  is locally the trace on  $M_{\mathbb{R}}$  of a Stein open set in any given complexification  $M_{\mathbb{C}}$  of  $M_{\mathbb{R}}$ .

The analogous result in the semi-analytic setting is easy to obtain because when  $f$  is a real analytic function, say near  $0$  in  $\mathbb{R}^n$ , the set  $\{f > 0\}$  is near  $0$  the trace on  $\mathbb{R}^n$  on the Stein open set  $\{\Re(f) > 0\}$  cut with a small open ball in  $\mathbb{C}^n$ .

We solve the subanalytic case of this problem using the rather deep following result (Theorem 2.1 below):

- a compact subanalytic set in  $\mathbb{R}^n$  may be defined as the zero set of a  $\mathcal{C}^2$  subanalytic function on  $\mathbb{R}^n$ .

The construction of the subanalytic Stein open subset we are looking for is then an easy consequence of H. Grauert's idea.

Let us mention without technical details that applications of our result arise naturally in the theory of sheaves on subanalytic sites, as it has been developed by L. Prelli in [14] (cf. [11] for the foundations of the theory of

ind-sheaves). It entails, for instance, that the subanalytic sheaf of tempered analytic functions on a real analytic manifold is concentrated in degree zero as in the classical case.

We conclude this article by computing one very simple example which is not semi-analytic in order to show that the subanalytic case is much more involved and also to explain to non specialists of subanalytic geometry (as we are) what are the ideas and tools hidden behind this construction.

We wish to thank Adam Parusinski for having pointed out to us a precise reference of Theorem 2.1, and the referee for asking us about the unbounded case.

## 2 Main results and proofs

We refer to [1], [3], [10] and [13] for the basic material on subanalytic geometry.

The following result is due to Bierstone, Milman and Pawlucki in a private letter to W. Schmid and K. Vilonen in 1995 (cf. [16]). We refer [4], C.11, for a proof in the more general setting of o-minimal structures.

**Theorem 2.1.** *Let  $A$  be a compact subanalytic set of  $\mathbb{R}^n$  and let  $p \in \mathbb{N}$  be given. Then there exists a  $\mathcal{C}^p$  subanalytic function  $f$  in  $\mathbb{R}^n$  such that  $A = f^{-1}(0)$ .*

**Remark 2.2.** Let  $U$  be a open ball in  $\mathbb{R}^n$  and  $Z$  a relatively compact subanalytic open set in  $U$ . Then there exists a  $\mathcal{C}^2$  subanalytic function  $g : \mathbb{R}^n \rightarrow \mathbb{R}^+$  with compact support in  $U$  such that

$$Z = \{x \in \mathbb{R}^n ; g(x) > 0\}.$$

Apply the previous theorem to  $\bar{U} \setminus Z$  and define  $g$  to be  $f$  on  $U$  and 0 on  $\mathbb{R}^n \setminus U$ . As  $U$  is subanalytic and  $f$  identically zero around  $\partial U$ , this function  $g$  satisfies the required properties.

Moreover, we can divide this function  $g$  by any given positive constant

without changing the set  $Z$ , so for each  $\varepsilon > 0$  we may assume that the Levi form of  $g$  is uniformly bounded on  $\mathbb{R}^n$  by  $\varepsilon \cdot \|z\|^2$ .

**Corollary 2.3.** *Let  $\Omega$  be a subanalytic open set in a real paracompact analytic manifold  $M_{\mathbb{R}}$ . Then, for any complexification  $M_{\mathbb{C}}$  of  $M_{\mathbb{R}}$ , and for any given smooth hermitian metric on the complex tangent bundle on  $M_{\mathbb{C}}$  there exists a subanalytic non negative real function  $f$  on  $M_{\mathbb{C}}$  of class  $\mathcal{C}^2$  such that*

$$\{f > 0\} \cap M_{\mathbb{R}} = \Omega$$

*and such that the Levi form of  $f$  is bounded by the given hermitian metric. Moreover,  $f$  can be chosen so that  $\text{Supp} f$  is contained in any given open set in  $M^{\mathbb{C}}$  containing the closed set  $\bar{\Omega}$ .*

*Proof.* For  $\epsilon > 0$ , let us denote  $B_{\epsilon}$  an open ball of  $\mathbb{R}^n$  of radius  $\epsilon$  and by  $B_{\epsilon}^{\mathbb{C}}$  the corresponding ball in  $\mathbb{C}^n$ .

For each  $p \in \bar{\Omega}$  (the closure of  $\Omega$ ) there exists two relatively compact open subanalytic neighbourhoods  $V \subset \subset U$  of  $p$  in  $M_{\mathbb{C}}$  and a complex analytic isomorphism  $\varphi$  defined in an open neighbourhood  $W$  of  $\bar{U}$  to an open ball  $B_{\epsilon}^{\mathbb{C}}$  such that  $\varphi(\bar{V})$  is the closed ball  $\bar{B}_{\epsilon/2}^{\mathbb{C}}$ , and  $\varphi$  is real on  $W \cap M_{\mathbb{R}}$ . In particular,  $\bar{V} \cap M_{\mathbb{R}} \subset U$  is a compact subanalytic subset, and  $\bar{U}$  is a compact subanalytic subset of  $W$ .

As  $M_{\mathbb{R}}$  is paracompact, we get a locally finite countable cover  $(U_i)_{i \in \mathbb{N}^*}$  of  $\bar{\Omega}$  such that the conditions above are satisfied. On each  $U_i$ , by the remark following the Theorem 2.1, we may choose a  $\mathcal{C}^2$  non negative subanalytic function  $f_i$  on  $M_{\mathbb{C}}$  with compact support in  $U_i$  whose non zero set is exactly  $V_i \cap \Omega$ , and such that its Levi form is bounded by  $h/2^i$  for any given hermitian metric  $h$  on  $M_{\mathbb{C}}$ . Then define  $f := \sum_{i=1}^{\infty} f_i$ . As this sum is locally finite, it clearly satisfies our requirements.

The last assertion follows by applying this construction in any open neighbourhood  $W$  of  $\bar{\Omega}$  in  $M_{\mathbb{C}}$  regarded as a complexification of  $W \cap M_{\mathbb{R}}$ .  
q.e.d.

**Theorem 2.4.** *Let  $\Omega$  be a subanalytic open set of a real paracompact analytic manifold  $M_{\mathbb{R}}$ . Then, given a complexification  $M_{\mathbb{C}}$  of  $M_{\mathbb{R}}$ , there exists a subanalytic Stein open subset  $\Omega_{\mathbb{C}}$  of  $M_{\mathbb{C}}$  such that*

$$(2.1) \quad \Omega = \Omega_{\mathbb{C}} \cap M_{\mathbb{R}}$$

*Proof.* Let  $n$  be the dimension of  $M_{\mathbb{R}}$ . By Grauert's Theorem 3, page 470 of [5], there exist a natural number  $N \in \mathbb{N}$  and a real analytic regular proper embedding  $\varphi$  of  $M_{\mathbb{R}}$  in the euclidean space  $\mathbb{R}^N$ . By complexification, one defines a holomorphic map  $\varphi_{\mathbb{C}}$  in a neighborhood  $V$  of  $M_{\mathbb{R}}$  in  $M_{\mathbb{C}}$  taking values in  $\mathbb{C}^N$ , such that  $\varphi_{\mathbb{C}}|_{M_{\mathbb{R}}} = \varphi$  and such that the rank of  $\varphi_{\mathbb{C}}$  is everywhere equal to  $n$ .

Note that the Levi form of the real analytic function

$$g(z_1, \dots, z_N) = \sum_{j=1}^N (\Im z_j)^2$$

is half the square norm in  $\mathbb{C}^N$ , hence  $g$  is strictly plurisubharmonic on  $\mathbb{C}^N$ . By the maximality of the rank of  $\varphi_{\mathbb{C}}$ , the function  $\varphi_{\mathbb{C}}^*(g)$  is also strictly plurisubharmonic on  $V$  and subanalytic (in fact analytic).

Fix now a smooth hermitian metric<sup>1</sup>  $h$  on  $T_{\mathbb{C}}V$  such that the Levi form of  $\varphi_{\mathbb{C}}^*(g)$  is bigger at each point than  $2h$ .

By Proposition 2.3, there exists a subanalytic  $\mathcal{C}^2$  non negative function  $f$  with support on  $V$  such that

$$\{f > 0\} \cap M_{\mathbb{R}} = \Omega$$

and such that the Levi form of  $f$  is bounded by  $h$ . So the Levi form of the  $\mathcal{C}^2$  subanalytic function

$$\psi := \varphi_{\mathbb{C}}^*(g) - f$$

is positive definite at each point of  $V$ . It follows that the open set

$$\Omega_{\mathbb{C}} = \{\psi < 0\} \cap V$$

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<sup>1</sup>for instance  $1/2$  of the Levi form of  $\varphi_{\mathbb{C}}^*(g)$  may be choose as Kähler form on  $V$ .

is (strongly 1-complete) Stein by Grauert's famous result and subanalytic in  $M_{\mathbb{C}}$  by construction.

Moreover, as we have  $\varphi_{\mathbb{C}}^*(g) = 0$  in  $M_{\mathbb{R}}$ , it follows that  $\Omega_{\mathbb{C}} \cap M_{\mathbb{R}} = \Omega$ .  
q.e.d.

### 3 Example: A strange four-leaved trefoil

Our aim is now to give an explicit construction of the function  $f$  in Theorem 2.1 in the case of one of the simplest example which is not semi-analytic. For that purpose we shall only use Łojasiewicz inequalities and Theorem 3.2 which are basic tools in subanalytic geometry. We think that this analysis will convince the reader of the strength and usefulness of Theorem 2.1 and that this tool is far from being elementary.

We shall need the following refinement of subanalyticity.

#### 3.1 Strong subanalyticity

For a continuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  to be subanalytic simply means that its graph is a subanalytic set in  $\mathbb{R}^n \times \mathbb{R}$ , but in the non continuous case we shall use a stronger assumption, in order to control the behaviour of the graph near points where  $f$  is not locally bounded. We restrict ourself to the context of the situation we need here.

**Definition 3.1.** Let  $\Omega \subset \subset \mathbb{R}^n$  a relatively compact subanalytic open set, and let

$$f : \Omega \rightarrow \mathbb{R}$$

be a continuous function. We shall say that  $f$  is **strongly subanalytic** if the function  $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by extending  $f$  by 0 on  $\mathbb{R}^n \setminus \Omega$  has a subanalytic graph in  $\mathbb{R}^n \times \mathbb{P}_1$ , where  $\mathbb{P}_1$  is the 1-dimensional projective space  $\mathbb{R} \cup \{\infty\}$ .

It is easy to see that such a condition implies that the growth of  $f$  near a boundary point in  $\partial\Omega$  has to be bounded by some power of the function  $d(x, \partial\Omega)$  thanks to Łojasiewicz inequalities ([1]).

Remark that if  $\tilde{f}$  is continuous this condition reduces to the usual subanalyticity of the graph of  $\tilde{f}$  in  $\mathbb{R}^n \times \mathbb{R}$ .

We shall need also the following theorem (cf.[10], Theorem (2.4)).

**Theorem 3.2.** *Let  $\Omega \subset\subset \mathbb{R}^n$  a relatively compact subanalytic open set, and let*

$$f : \Omega \rightarrow \mathbb{R}$$

*be a  $\mathcal{C}^1$  function which is strongly subanalytic. Then any partial derivative of  $f$  in  $\Omega$  is also strongly subanalytic.*

Since, in Definition 3.1, the continuity of  $\tilde{f}$  just means that  $f(x)$  goes to 0 when  $x \in \Omega$  goes to the boundary  $\partial\Omega$ , using Łojasiewicz inequalities we easily obtain the following corollary:

**Corollary 3.3.** *In the situation of the previous theorem, assume that  $\tilde{f}$  is continuous. Then there exists an integer  $N_1$  such that  $\tilde{f}^{N_1}$  is  $\mathcal{C}^1$  on  $\mathbb{R}^n$  and subanalytic.*

Now applying again the ideas of the previous corollary we finally obtain:

**Corollary 3.4.** *In the situation of the previous corollary there exists an integer  $N_2$  such that  $\tilde{f}^{N_2}$  is  $\mathcal{C}^2$  on  $\mathbb{R}^n$  and subanalytic.*

**Remark 3.5.** As the reader can see in view of the preceding results, the remaining and non trivial step to prove the existence of a subanalytic  $\mathcal{C}^2$  function which vanishes exactly on  $\mathbb{R}^n \setminus \Omega$  as stated in Theorem 2.1, is to show the existence of a  $\mathcal{C}^2$  strictly positive (strongly) subanalytic function  $f$  on  $\Omega$  which vanishes at the boundary. The natural candidate is, of course, the function  $x \mapsto d(x, \partial\Omega)$ . But all conditions are satisfied excepted smoothness. And the non smoothness points may go to the boundary. If

one tries to use the "desingularization theorem" of H. Hironaka to solve this problem, a new difficulty comes then because the jacobian of the modification may vanish inside  $\Omega$  and not only on some points in  $\partial\Omega$ .

### 3.2 Example

Let us consider the analytic map  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by

$$F(x, y, z) = (y \cdot (e^x - 1) + x^2 + y^2 + z^2 - \varepsilon^2, y \cdot (e^{x \cdot \sqrt{2}} - 1), y \cdot (e^{x \cdot \sqrt{3}} - 1)).$$

Denote  $\Omega$  the interior of the image  $\tilde{\Omega}$  of the compact ball  $\bar{B}_3(0, \varepsilon)$ . Let us start by showing that the image by  $F$  of the sphere  $S_\varepsilon$  (the boundary of  $\bar{B}(0, \varepsilon)$ ), is a subanalytic compact subset of  $\mathbb{R}^3$  which is not semi-analytic in the neighborhood of  $(0, 0, 0)$ . This example is extracted from [8] (example I.6).

**Lemma 3.6.** *The compact  $F(S_\varepsilon)$  is not semi-analytic in the neighbourhood of the origin.*

*Proof.* Since this compact set has an empty interior, if it is semi-analytic in a neighbourhood of the origin, there shall exist an analytic function  $f : U \rightarrow \mathbb{R}$  on a ball  $U$  centered in  $0$ , non identically zero, such that  $f^{-1}(0)$  contains  $U \cap F(S_\varepsilon)$ . Let

$$f = \sum_{m \geq m_0} P_m$$

be the Taylor series of  $f$  at the origin, which we may assume to be convergent in  $U$  provided that  $U$  is small enough. We shall assume that the homogeneous polynomial  $P_{m_0}$  is not identically zero. Hence, considering  $(x, y, z) \in S_\varepsilon$  close enough to  $(0, 0, \varepsilon)$ , the definition of  $F$  entails the equality

$$0 \equiv \sum_{m \geq m_0} y^m \cdot P_m((e^x - 1), (e^{x \cdot \sqrt{2}} - 1), (e^{x \cdot \sqrt{3}} - 1))$$

which holds for  $(x, y) \in \mathbb{R}^2$  close enough to  $(0, 0)$ . We conclude that  $P_{m_0}((e^x - 1), (e^{x \cdot \sqrt{2}} - 1), (e^{x \cdot \sqrt{3}} - 1))$  is identically zero for  $x$  in a neighbourhood of  $0$ . Hence this analytic function vanishes identically on  $\mathbb{R}$ .



The behaviour at infinity of this function easily entails<sup>2</sup> that we must have  $P_{m_0} \equiv 0$ , which gives a contradiction. q.e.d.

We shall now describe the open set  $\Omega$ . Let us remark that the jacobian of  $F$  is given by

$$J(F)(x, y, z) = 2yz \cdot ((\sqrt{2} - \sqrt{3}) \cdot e^{x \cdot (\sqrt{2} + \sqrt{3})} - \sqrt{2} \cdot e^{x \cdot \sqrt{2}} + \sqrt{3} \cdot e^{x \cdot \sqrt{3}})$$

and for  $\varepsilon$  small enough, it doesn't vanish on  $\{x \cdot y \cdot z = 0\}$  within the ball  $\bar{B}_3(0, \varepsilon)$ . Indeed, the brackets give an analytic function of a single variable  $x$ ; hence it has an isolated zero in  $x = 0$ . The image of  $\{x \cdot y = 0\} \cap \bar{B}_3(0, \varepsilon)$  by  $F$  is  $[-\varepsilon^2, 0] \times \{(0, 0)\}$  which is contained in<sup>3</sup> the boundary of  $\tilde{\Omega}$ . The image of  $\{z = 0\}$  is more complicated to describe.

Let us now consider the analytic morphism  $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$G(x, y) := (y \cdot (e^{x \cdot \sqrt{2}} - 1), y \cdot (e^{x \cdot \sqrt{3}} - 1)).$$

Let us denote by  $\Gamma$  the image by  $G$  of the ball  $\bar{B}_2(0, \varepsilon)$  of  $\mathbb{R}^2$ . If  $(v, w) \in \Gamma \setminus \{(0, 0)\}$  then the fiber  $G^{-1}(v, w)$  is reduced to a single point (for  $\varepsilon$  small enough). In fact we must have  $v \cdot w \neq 0$  and

$$\frac{(e^{x \cdot \sqrt{2}} - 1)}{(e^{x \cdot \sqrt{3}} - 1)} = \frac{v}{w} = \frac{\sqrt{2}}{\sqrt{3}} \cdot h(x)$$

whenever  $h \in \mathbb{C}\{x\}$  converges for  $|x| < 2\pi/\sqrt{3}$  et verifies  $h(0) = 1$  and  $h'(0) = (\sqrt{2} - \sqrt{3})/2$ ; these equations determine a unique  $x \in [-\varepsilon, \varepsilon]$ , for  $\varepsilon \ll 1$ , and hence a unique  $y$ . Remark that for  $x$  in a neighbourhood of 0, we have  $v/w$  close to  $\sqrt{2}/\sqrt{3}$ . Therefore  $\Gamma$  doesn't approach  $(0, 0)$  other than along that direction.

The fiber in  $(0, 0)$  of  $G$  is the curve  $\{x \cdot y = 0\} \cap \bar{B}_2(0, \varepsilon)$ .

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<sup>2</sup> This is equivalent to prove the algebraic independency of the functions  $(e^x - 1), (e^{x \cdot \sqrt{2}} - 1), (e^{x \cdot \sqrt{3}} - 1)$ .

<sup>3</sup> See the description of  $\Gamma$  near  $(0, 0)$  given below

Remark that the points in the sphere  $\{x^2 + y^2 = \varepsilon^2\}$  are mapped on the boundary of  $\Gamma$ . Indeed, for those who lie on  $\{x.y = 0\}$  their image is the origin. Otherwise, for each of such points not mapped on the origin, the jacobian of  $G$  would vanish and the boundary of  $\bar{B}_2(0, \varepsilon)$  would be mapped on the boundary of  $\Gamma$  in its neighbourhood.

Hence, any point of the interior  $\Gamma'$  of  $\Gamma$  is the image by  $G$  of some point in  $B_2(0, \varepsilon) \setminus \{x.y = 0\}$ .

We shall denote by  $\varphi : \Gamma \setminus \{(0, 0)\} \rightarrow \mathbb{R}$  the subanalytic function<sup>4</sup> given by  $\varphi(v, w) = ||G^{-1}(v, w)||^2$ , in other words, the composition of  $G^{-1}$  with the square of the euclidean norm in  $\mathbb{R}^2$ .

We shall denote by  $\psi : \Gamma \setminus \{(0, 0)\} \rightarrow \mathbb{R}$  the subanalytic function defined by setting  $\psi(v, w) = y.(e^x - 1)$  where  $G^{-1}(v, w) = (x, y)$ , and we set

$$\begin{aligned}\Delta^+ &:= \{(\psi(v, w), v, w), \quad \text{for } (v, w) \in \Gamma \setminus \{(0, 0)\}\} \\ \Delta^- &:= \{(\psi(v, w) + \varphi(v, w) - \varepsilon^2, v, w), \quad \text{for } (v, w) \in \Gamma \setminus \{(0, 0)\}\} \\ \Delta^0 &:= [-\varepsilon^2, 0] \times \{(0, 0)\}\end{aligned}$$

Note that

$$\Delta^+ \cap \Delta^- = \{(u, v, w) \in \mathbb{R} \times (\Gamma \setminus \{(0, 0)\}) / u = \psi(v, w) \quad \text{and} \quad \varphi(v, w) = \varepsilon^2\}$$

is the graph of the restriction of  $\psi$  to  $\partial\Gamma \setminus \{(0, 0)\}$ .

We have now the following description of  $\tilde{\Omega}$  and of its interior  $\Omega$ .

**Lemma 3.7.** *One has  $\partial\tilde{\Omega} = \Delta^+ \cup \Delta^- \cup \Delta^0$ . The interior  $\Omega$  is the open set*

$$\Omega = \{(u, v, w) \in \mathbb{R} \times \Gamma' / \psi(v, w) + \varphi(v, w) - \varepsilon^2 < u < \psi(v, w)\}$$

where  $\Gamma'$  denotes the interior of  $\Gamma$ .

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<sup>4</sup> The graph of  $G^{-1} : \Gamma \setminus \{(0, 0)\} \rightarrow \bar{B}_2(0, \varepsilon) \setminus \{x.y = 0\}$  is the same as that the graph of  $G : \bar{B}_2(0, \varepsilon) \setminus \{x.y = 0\} \rightarrow \Gamma \setminus \{(0, 0)\}$ .

*Proof.* Let  $(u, v, w) \in \tilde{\Omega}$ . If  $v.w = 0$  then  $x.y = 0$  and  $v = w = 0$ , and  $u = x^2 + y^2 + z^2 - \varepsilon^2$  belongs to  $[-\varepsilon^2, 0]$  which is contained in  $\Delta^0$ . Since the projection of  $\Omega$  on  $\mathbb{R}^2$  is an open set contained in  $\Gamma$ , hence in  $\Gamma'$ , the point  $(u, v, w)$  does not belong to  $\Omega$ . Let us now exclude this case.

We have a point  $(x, y, z) \in \bar{B}_3(0, \varepsilon)$  such that  $F(x, y, z) = (u, v, w)$ , with  $x.y \neq 0$ . Then  $(x, y) \in \bar{B}_2(0, \varepsilon) \setminus \{x.y = 0\}$  and  $G(x, y) = (v, w)$  is not  $(0, 0)$ . Since  $\varphi(v, w) = x^2 + y^2$  we have

$$u = \psi(v, w) + \varphi(v, w) + z^2 - \varepsilon^2$$

where  $z \in [-\varepsilon, \varepsilon]$  is, up to a sign, determined by this equation. We conclude that the inequalities

$$(3.1) \quad \psi(v, w) + \varphi(v, w) - \varepsilon^2 \leq u \leq \psi(v, w)$$

hold on  $\tilde{\Omega}$ . We have to check that  $\partial\tilde{\Omega} \setminus \Delta^0$  is exactly described by the equality

$$(3.2) \quad (u - \psi(v, w) - \varphi(v, w) + \varepsilon^2)(\psi(v, w) - u) = 0.$$

Since the projection on  $\mathbb{R}^2$  is open, if  $(v, w) \notin \Gamma'$  then it must lie in the boundary of  $\Omega$ . It suffices to prove that for  $(v, w) \in \Gamma'$  the equality above implies that  $(v, w)$  is in the boundary. This is clear because near any  $(u, v, w)$  of  $\Omega$  one can find  $\delta > 0$  such that  $]u - \delta, u + \delta[ \times (v, w)$  is contained in  $\Omega$ , which is not possible by the inequalities (3.1) in a point where the equality (3.2) is satisfied.

Hence it is sufficient to prove that  $\tilde{\Omega} \setminus \Delta^0$  is the set of points  $(u, v, w)$  in  $\mathbb{R} \times (\Gamma \setminus \{(0, 0)\})$  satisfying the inequalities (3.1). Indeed, any choice of  $(v, w) \in \Gamma \setminus \{(0, 0)\}$  gives a unique point  $(x, y) \in \bar{B}_2(0, \varepsilon)$  such that  $G(x, y) = (v, w)$  and the inequalities (3.1) entail that we can find at least a  $z \in \mathbb{R}$  such that  $z^2 = u - \psi(v, w) - \varphi(v, w) + \varepsilon^2$  and that  $\varphi(v, w) + z^2 \leq \varepsilon^2$ . Note that if  $u = \psi(v, w) + \varphi(v, w) - \varepsilon^2$  we will have  $z = 0$ . Therefore, the boundary  $\Delta^-$  corresponds to the image of  $\bar{B}_3(0, \varepsilon) \cap \{z = 0\} \setminus \Delta^0$ . Similarly the equality  $u = \psi(v, w)$  corresponds to the image of the sphere  $\{x^2 + y^2 + z^2 = \varepsilon^2\}$  deprived of  $\Delta^0$ . q.e.d.

Let us now consider the function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^+$  defined as follows:

- For  $(u, v, w) \in \Omega$  one sets  

$$f(u, v, w) = (\psi(v, w) - u)(u - \psi(v, w) - \varphi(v, w) + \varepsilon^2)$$
- For  $(u, v, w) \notin \Omega$  one sets  $f(u, v, w) = 0$ .

Note that  $f$  is strictly positive on  $\Omega$  by Lemma 3.7, and that it is analytic on the complement of  $\partial\Omega$ , since the functions  $\varphi$  and  $\psi$  are analytic on  $\Gamma'$ . Moreover  $f$  is bounded.

Let us now define  $\tilde{f}(u, v, w) = f(u, v, w).v^2.w^2$ .

**Lemma 3.8.** *The function  $\tilde{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^+$  is subanalytic and continuous, it satisfies*

$$\Omega = \{(u, v, w) \in \mathbb{R}^3 \mid \tilde{f}(u, v, w) > 0\}$$

and it is  $\mathcal{C}^\infty$  on  $\mathbb{R}^3 \setminus \partial\Omega$ .

*Proof.* First we prove that  $f$  is subanalytic<sup>5</sup>. Since its graph is the union of the graph of its restriction to  $\Omega$  and the set  $(\mathbb{R}^3 \setminus \Omega) \times \{0\}$  which is subanalytic,  $\Omega$  being an open subanalytic set of  $\mathbb{R}^3$ , it is sufficient to prove that the graph of the restriction of  $f$  to  $\Omega$  is subanalytic.

Let us consider the polynomial morphism  $h : \mathbb{R}^3 \rightarrow \mathbb{R}$  given by

$$h(x, y, z) = (\varepsilon^2 - (x^2 + y^2 + z^2)).z^2$$

and denote by  $X, X_1, X_2$  the graph of the restriction of  $h$  respectively to  $\bar{B}_3(0, \varepsilon), \partial B_3(0, \varepsilon), \bar{B}_3(0, \varepsilon) \cap \{x.y = 0\}$  and  $Y, Y_1, Y_2$  the respective images of these graphs by the morphism  $F \times id : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}$ .

Let us prove that the graph of the restriction of  $f$  to  $\Omega$  is equal to  $Y \setminus (Y_1 \cup Y_2)$ . Indeed, for  $(u, v, w) \in \Omega$ , if  $(x, y, z) \in \bar{B}_3(0, \varepsilon)$  verifies  $F(x, y, z) = (u, v, w)$ , we get  $\varphi(v, w) = x^2 + y^2, \psi(v, w) = y.(e^x - 1)$  and  $u = \psi(v, w) + \varphi(v, w) + z^2 - \varepsilon^2$ .

One sees that  $f(u, v, w) = (\varepsilon^2 - (x^2 + y^2 + z^2)).z^2$ . To finish, it is enough to

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<sup>5</sup>As pointed by the referee, this fact is consequence of basic stability properties of subanalytic functions. We give a direct proof for non specialists.

note that the points of  $F(\bar{B}_3(0, \varepsilon) \cap \{x.y = 0\})$  and of  $F(\partial B_3(0, \varepsilon))$  are never in  $\Omega$ . Hence  $\tilde{f}$  is subanalytic.

Let us show that it is continuous along  $\partial\Omega$ , since it is  $\mathcal{C}^\infty$  on  $\mathbb{R}^3 \setminus \partial\Omega$ . Let  $(u_0, v_0, w_0) \in \partial\Omega$ . First assume that  $(u_0, v_0, w_0)$  belongs to  $\Delta^+$ . Then  $u_0 = \psi(v_0, w_0)$ , in other words, we get the image by  $F$  of a point  $(x, y, z) \in \partial B_3(0, \varepsilon) \setminus \{x.y = 0\}$ . Hence the limit of  $(u - \psi(v, w))$  when  $(u, v, w) \in \Omega$  tends to  $(u_0, v_0, w_0)$  is zero. As the functions  $\psi$  and  $\varphi$  are bounded on  $\Omega$ , the limit of  $\tilde{f}$  is zero in such a point.

If  $(u_0, v_0, w_0) \in \Delta^-$ , then we have the image of a point in

$$(\bar{B}_3(0, \varepsilon) \cap \{z = 0\}) \setminus \{x.y = 0\}.$$

Since the function  $\psi$  is bounded on  $\Gamma$  the limit of  $f$  in such a point is zero, and so it is for  $\tilde{f}$ .

If  $(u_0, v_0, w_0) \in \Delta^0$  then we have  $v_0 = w_0 = 0$  and the function  $f$  is bounded, hence  $\tilde{f}$  tends to 0 in such a point.

Let us finally show that  $\Omega$  is the set where  $\tilde{f}$  is strictly positive. It is sufficient to check that  $v.w \neq 0$  on  $\Omega$ . But  $v.w = 0$  entails  $x.y = 0$  and so  $v = w = 0$  and  $u = x^2 + y^2 + z^2 - \varepsilon^2$ , in other words,  $(u, v, w) \in [-\varepsilon^2, 0] \times (0, 0) = \Delta^0$ . Hence such  $(v, w)$  belongs to  $\partial\Omega$ .

We have now constructed a subanalytic function  $\tilde{f}$  on  $\mathbb{R}^3$  which is continuous and strictly positive exactly on  $\Omega \subset \subset \mathbb{R}^3$ . By Corollary 3.4 there exists a positive integer  $N$  such that  $\tilde{f}^N$  is of class  $\mathcal{C}^2$ . Then one gets a Stein open subanalytic set of  $\mathbb{C}^3$  which cuts  $\mathbb{R}^3$  exactly on  $\Omega$  as in the general proof of Theorem 2.4. q.e.d.

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